On the Frobenius integrability of certain holomorphic p-forms

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Dedicated to Professor Hans Grauert, on the occasion of his 70th birthday

Abstract. The goal of this note is to exhibit the integrability properties (in the sense of the Frobenius theorem) of holomorphic p-forms with values in certain line bundles with semi-negative curvature on a compact Kähler manifold. There are in fact very strong restrictions, both on the holomorphic form and on the curvature of the semi-negative line bundle. In particular, these observations provide interesting information on the structure of projective manifolds which admit a contact structure: either they are Fano manifolds or, thanks to results of Kebekus-Peternell-Sommese-Wisniewski, they are biholomorphic to the projectivization of the cotangent bundle of another suitable projective manifold.

1. Main results

Recall that a holomorphic line bundle L on a compact complex manifold is said to be *pseudo-effective* if $c_1(L)$ contains a closed positive $(1,1)$-current T, or equivalently, if L possesses a (possibly singular) hermitian metric h such that the curvature current $T = \Theta_h(L) = -i\partial\overline{\partial}\log h$ is nonnegative. If X is projective, L is pseudo-effective if and only if $c_1(L)$ belongs to the closed cone of $H^{1,1}_{\mathbb{R}}(X)$ generated by classes of effective divisors (see [Dem90, 92]). Our main result is

Main Theorem. *Let X be a compact Kähler manifold. Assume that there exists a pseudo-effective line bundle L on X and a nonzero holomorphic section $\theta \in H^0(X, \Omega^p_X \otimes L^{-1})$, where $0 \leqslant p \leqslant n = \dim X$. Let $\mathcal{S}_\theta$ be the coherent subsheaf of germs of vector fields ξ in the tangent sheaf T_X, such that the contraction $i_\xi\theta$ vanishes. Then $\mathcal{S}_\theta$ is integrable, namely $[\mathcal{S}_\theta, \mathcal{S}_\theta] \subset \mathcal{S}_\theta$, and L has flat curvature along the leaves of the (possibly singular) foliation defined by $\mathcal{S}_\theta$.*

Before entering into the proof, we discuss several consequences. If $p = 0$ or $p = n$, the result is trivial (with $\mathcal{S}_\theta = T_X$ and $\mathcal{S}_\theta = 0$, respectively). The most interesting case is $p = 1$.

Corollary 1. *In the above situation, if the line bundle $L \to X$ is pseudo-effective and $\theta \in H^0(X, \Omega^1_X \otimes L^{-1})$ is a nonzero section, the subsheaf $\mathcal{S}_\theta$ defines a holomorphic foliation of codimension 1 in X, that is, $\theta \wedge d\theta = 0$.*

We now concentrate ourselves on the case when X is a *contact manifold*, i.e. $\dim X = n = 2m+1$, $m \geqslant 1$, and there exists a form $\theta \in H^0(X, \Omega^1_X \otimes L^{-1})$, called the *contact form*, such that $\theta \wedge (d\theta)^m \in H^0(X, K_X \otimes L^{-m-1})$ has no zeroes.

Then $\mathcal{S}_\theta$ is a codimension 1 locally free subsheaf of T_X and there are dual exact sequences

$$0 \to L \to \Omega^1_X \to \mathcal{S}^\star_\theta \to 0, \qquad 0 \to \mathcal{S}_\theta \to T_X \to L^\star \to 0.$$

The subsheaf $\mathcal{S}_\theta \subset T_X$ is said to be the *contact structure* of X. The assumption that $\theta\wedge(d\theta)^m$ does not vanish implies that $K_X \simeq L^{m+1}$. In that case, the subsheaf is not integrable, hence L and K_X cannot be pseudo-effective.

Corollary 2. *If X is a compact Kähler manifold admitting a contact structure, then K_X is not pseudo-effective, in particular the Kodaira dimension $\kappa(X)$ is equal to $-\infty$.*

The fact that $\kappa(X) = -\infty$ had been observed previously by Stéphane Druel [Dru98]. In the projective context, the minimal model conjecture would imply (among many other things) that the conditions $\kappa(X) = -\infty$ and "K_X non pseudo-effective" are equivalent, but a priori the latter property is much stronger (and in large dimensions, the minimal model conjecture still seems far beyond reach!)

Corollary 3. *If X is a compact Kähler manifold with a contact structure and with second Betti number $b_2 = 1$, then K_X is negative, i.e., X is a Fano manifold.*

Actually the Kodaira embedding theorem shows that the Kähler manifold X is projective if $b_2 = 1$, and then every line bundle is either positive, flat or negative. As K_X is not pseudo-effective it must therefore be negative. In that direction, Boothby [Boo61], Wolf [Wol65] and Beauville [Bea98] have exhibited a natural construction of contact Fano manifolds. Each of the known examples is obtained as a homogeneous variety which is the unique closed orbit in the projectivized (co)adjoint representation of a simple algebraic Lie group. Beauville's work ([Bea98], [Bea99]) provides strong evidence that this is the complete classification in the case $b_2 = 1$.

We now come to the case $b_2 \geqslant 2$. If Y is an arbitrary compact Kähler manifold, the bundle $X = P(T^\star_Y)$ of hyperplanes of T_Y has a contact structure associated with the line bundle $L = \mathcal{O}_X(-1)$. Actually, if $\pi : X \to Y$ is the canonical projection, one can define a contact form $\theta \in H^0(X, \Omega^1_X \otimes L^{-1})$ by setting

$$\theta(x) = \theta(y, [\xi]) = \xi^{-1}\pi^\star\xi = \xi^{-1}\sum_{1\leqslant j\leqslant p} \xi_j dy_j, \qquad p = \dim Y,$$

at every point $x = (y, [\xi]) \in X$, $\xi \in T^\star_{Y,y} \smallsetminus \{0\}$ (observe that $\xi \in L_x = \mathcal{O}_X(-1)_x$). Morever $b_2(X) = 1 + b_2(Y) \geqslant 2$. Conversely, Kebekus, Peternell, Sommese and Wiśniewski [KPSW] have recently shown that every projective algebraic manifold X such that

(i) X has a contact structure,

(ii) $b_2 \geqslant 2$,

(iii) K_X is not nef (numerically effective)

is of the form $X = P(T^\star_Y)$ for some projective algebraic manifold Y. However, the condition that K_X is not nef is implied by the fact that K_X is not pseudo-effective. Hence we get

Corollary 4. *If X is a contact projective manifold with $b_2 \geqslant 2$, then X is a projectivized hyperplane bundle $X = P(T^\star_Y)$ associated with some projective manifold Y.*

The Kähler case of corollary 4 is still unsolved, as the proof of [KPSW] heavily relies on Mori theory (and, unfortunately, the extension of Mori theory to compact Kähler manifolds remains to be settled ...).

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2. Proof of the Main Theorem

In some sense, the proof is just a straightforward integration by parts, but there are slight technical difficulties due to the fact that we have to work with singular metrics.

Let X be a compact Kähler manifold, ω the Kähler metric, and let L be a pseudo-effective line bundle on X. We select a hermitian metric h on L with nonnegative curvature current $\Theta_h(L) \geqslant 0$, and let φ be the plurisubharmonic weight of the metric h in any local trivialisation $L_{|U} \simeq U \times \mathbb{C}$. In other words, we have

$$\|\xi\|_h^2 = |\xi|^2 e^{-\varphi(x)}, \qquad \|\xi^\star\|_{h^\star}^2 = |\xi^\star|^2 e^{\varphi(x)}$$

for all $x \in U$ and $\xi \in L_x$, $\xi^\star \in L^{-1}$. We then have a Chern connection $\nabla = \partial_{h^\star} + \overline{\partial}$ acting on all (p,q)-forms f with values in L^{-1}, given locally by

$$\partial_\varphi f = e^{-\varphi}\partial(e^\varphi f) = \partial f + \partial\varphi \wedge f$$

in every trivialization $L_{|U}$. Now, assume that there is a holomorphic section $\theta \in H^0(X, \Omega_X^p \otimes L^{-1})$, i.e., a $\overline{\partial}$-closed $(p,0)$ form θ with values in L^{-1}. We compute the global L^2 norm

$$\int_X \{\partial_{h^\star}\theta, \partial_{h^\star}\theta\}_{h^\star} \wedge \omega^{n-p-1} = \int_X e^\varphi \partial_\varphi\theta \wedge \overline{\partial_\varphi\theta} \wedge \omega^{n-p-1}$$

where $\{\ ,\ \}_{h^\star}$ is the natural sesquilinear pairing sending pairs of L^{-1}-valued forms of type (p,q), (r,s) into $(p+s, q+r)$ complex valued forms. The right hand side is of course only locally defined, but it explains better how the forms are calculated, and also all local representatives glue together into a well defined global form; we will therefore use the latter notation as if it were global. As

$$d\big(e^\varphi\theta \wedge \overline{\partial_\varphi\theta} \wedge \omega^{n-p-1}\big) = e^\varphi\partial_\varphi\theta \wedge \overline{\partial_\varphi\theta} \wedge \omega^{n-p-1} + (-1)^p e^\varphi \theta \wedge \overline{\overline{\partial}\partial_\varphi\theta} \wedge \omega^{n-p-1}$$

and $\overline{\partial}\partial_\varphi\theta = \overline{\partial}\partial\varphi\wedge\theta$, an integration by parts via Stokes theorem yields

$$\int_X e^\varphi \partial_\varphi\theta\wedge\overline{\partial_\varphi\theta}\wedge\omega^{n-p-1} = -(-1)^p\int_X e^\varphi\partial\overline{\partial}\varphi\wedge\theta\wedge\overline{\theta}\wedge\omega^{n-p-1}.$$

These calculations need a word of explanation, since φ is in general singular. However, it is well known that the $i\partial\overline{\partial}$ of a plurisubharmonic function is a closed positive current, in particular

$$i\partial\overline{\partial}(e^\varphi) = e^\varphi(i\partial\varphi\wedge\overline{\partial}\varphi + i\partial\overline{\partial}\varphi)$$

is positive and has measure coefficients. This shows that $\partial\varphi$ is L^2 with respect to the weight e^φ, and similarly that $e^\varphi\partial\overline{\partial}\varphi$ has locally finite measure coefficients. Moreover, the results of [Dem92] imply that there is a decreasing sequence of metrics $h_\nu^\star$ and corresponding weights $\varphi_\nu\downarrow\varphi$, such that $\Theta_{h_\nu}\geqslant -C\omega$ with a fixed constant $C>0$ (this claim is in fact much weaker than the results of [Dem92], and easy to prove e.g. by using convolutions in suitable coordinate patches and a standard gluing technique). Now, the results of Bedford-Taylor [BT76, BT82] applied to the uniformly bounded functions $e^{c\varphi_\nu}$, $c>0$, imply that we have local weak convergence

$$e^{\varphi_\nu}\partial\overline{\partial}\varphi_\nu\to e^\varphi\partial\overline{\partial}\varphi,\quad e^{\varphi_\nu}\partial\varphi_\nu\to e^\varphi\partial\varphi,\quad e^{\varphi_\nu}\partial\varphi_\nu\wedge\overline{\partial}\varphi_\nu\to e^\varphi\partial\varphi\wedge\overline{\partial}\varphi,$$

possibly afting adding $C'|z|^2$ to the φ_ν's to make them plurisubharmonic. This is enough to justify the calculations. Now, we take care of signs, using the fact that $i^{p^2}\theta\wedge\overline{\theta}\geqslant 0$ whenever θ is a $(p,0)$-form. Our previous equality can be rewritten

$$\int_X e^\varphi\, i^{(p+1)^2}\partial_\varphi\theta\wedge\overline{\partial_\varphi\theta}\wedge\omega^{n-p-1} = -\int_X e^\varphi\, i\partial\overline{\partial}\varphi\wedge i^{p^2}\theta\wedge\overline{\theta}\wedge\omega^{n-p-1}.$$

Since the left hand side is nonnegative and the right hand side is nonpositive, we conclude that $\partial_\varphi\theta=0$ almost everywhere, i.e. $\partial\theta = -\partial\varphi\wedge\theta$ almost everywhere. The formula for the exterior derivative of a p-form reads

$$\begin{aligned} d\theta(\xi_0,\ldots,\xi_p) &= \sum_{0\leqslant j\leqslant p}(-1)^j\xi_j\cdot\theta(\xi_0,\ldots,\widehat{\xi_j},\ldots,\xi_p)\\ &\quad+\sum_{0\leqslant j<k\leqslant p}(-1)^{j+k}\theta([\xi_j,\xi_k],\xi_0,\ldots,\widehat{\xi_j},\ldots,\widehat{\xi_k},\ldots,\xi_p).\end{aligned}\tag{$\star$}$$

If two of the vector fields – say ξ_0 and ξ_1 – lie in $\mathcal{S}_\theta$, then

$$d\theta(\xi_0,\ldots,\xi_p) = -(\partial\varphi\wedge\theta)(\xi_0,\ldots,\xi_p) = 0$$

and all terms in the right hand side of $(\star)$ are also zero, except perhaps the term $\theta([\xi_0,\xi_1],\xi_2,\ldots,\xi_p)$. We infer that this term must vanish. Since this is true for

arbitrary vector fields $\xi_2, \ldots, \xi_p$, we conclude that $[\xi_0, \xi_1] \in \mathcal{S}_\theta$ and that $\mathcal{S}_\theta$ is integrable.

The above arguments also yield strong restrictions on the hermitian metric h. In fact the equality $\partial\theta = -\partial\varphi \wedge \theta$ implies $\partial\overline{\partial}\varphi \wedge \theta = 0$ by taking the $\overline{\partial}$. Fix a smooth point in a leaf of the foliation, and local coordinates $(z_1, \ldots, z_n)$ such that the leaves are given by $z_1 = c_1, \ldots, z_r = c_r$ ($c_i =$ constant), in a neighborhood of that point. Then $\mathcal{S}_\theta$ is generated by $\partial/\partial z_{r+1}, \ldots, \partial/\partial z_n$, and θ depends only on $dz_1, \ldots, dz_r$. This implies that $\partial^2\varphi/\partial z_j \partial\overline{z}_k = 0$ for $j, k > r$, in other words (L, h) has flat curvature along the leaves of the foliation. The main theorem is proved.

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